Polynomial expansion beats calculator display limits

by Charles Lotterman
Northrop Corp., Electronics Div., Hawthorne, Calif.

Occasionally, when you're multiplying or dividing two large numbers, you will exceed the display capacity of your calculator—even if you have a machine as sophisticated as Hewlett-Packard's HP-45, which rounds off the answer. But, by taking advantage of the way polynomials are multiplied or divided, you can get around this problem.

Any number can be expanded as a polynomial whose base is 1,000. For example, the number 123,456,789 can be written as:

$$123 \times 1{,}000^2 + 456 \times 1{,}000^1 + 789 \times 1{,}000^0$$

Now this number can be manipulated as a polynomial, with the three-digit significant figures of the number being treated as the coefficients of the polynomial.

To multiply two such polynomials:

■ Multiply each three-digit group of one number by each three-digit group of the other number in an orderly manner. (Your calculator's constant storage capability will be convenient to use during this operation.) For each multiplication, the digits that fall to the left of the three least-significant digits are carried into the next higher-order term.

■ Sum the three-digit terms that produce the corresponding power of 1,000, including all the carry factors from the lower-order terms.

■ Arrange the results in ascending order of powers of 1,000 to obtain the answers.

As an illustration of this technique, let's multiply 123,456,789 by itself. The carry terms will be enclosed by parentheses. The problem is:

$$[123\ 456\ 789] \times [123\ 456\ 789]$$

First, each three-digit group of the multiplicand is multiplied by the least-significant three digits of the multiplier:

$$789 \times 789 = (622)\ 521$$
$$456 \times 789 = (359)\ 784$$
$$123 \times 789 = (097)\ 047$$

Then, each three-digit group of the multiplicand is multiplied by the next-most-significant three digits of the multiplier:

$$789 \times 456 = (359)\ 784$$
$$456 \times 456 = (207)\ 936$$
$$123 \times 456 = (056)\ 088$$

Finally, each three-digit group of the multiplicand is multiplied by the most-significant three digits of the multiplier:

$$789 \times 123 = (097)\ 047$$
$$456 \times 123 = (056)\ 088$$
$$123 \times 123 = (015)\ 129$$

The results of each of these multiplications are arranged so that the three-digit groups belonging to the same power of 1,000 can be added together:

			123	456	789
		×	123	456	789
				(622)	521
			(359)	784	
		(097)	047		
			(359)	784	
		(207)	936		
	(056)	088			
		(097)	047		
	(056)	088			
(015)	129				
		(1)	(2)		
15	241	578	750	190	521

The answer, therefore, is: 15,241,578,750,190,521.

A similar technique can be used for division:

■ Set up the numbers in the format used for long division.

■ Perform a trial division using your calculator's divide function.

■ Round the results to a three-digit integer and multiply by the divisor.

■ Subtract the results of the multiplication from the dividend. The high-order term of the resulting polynomial must be zero.

■ Continue this process—dividing, multiplying, and subtracting, as in long division—until you obtain the desired number of places for the quotient.

■ Sum the results for the answer.

A numerical example will make the procedure clearer. We will divide 123,456,000 by 456,000. To keep the computations neat, let X = 1,000. The problem is:

$$\frac{123(X^2)+456(X^1)+000(X^0)+000(X^{-1})}{456(X^1)}$$

The trial division produces:

$$\frac{123{,}456{,}000}{456{,}000} = 270\,(X^0)$$

Proceed now as in long division. Multiply:

$$270(X^0) \times 456(X^1) = 123(X^2)+120(X^1)$$

Subtract:

$$\begin{array}{l} 123(X^2)+456(X^1)+000(X^0) \\ -\,123(X^2)+120(X^1) \\ \hline 000(X^2)+336(X^1)+000(X^0) \end{array}$$

Divide:

$$\frac{336(X^1)+000(X^0)}{456(X^1)} = 737(X^{-1})$$

Multiply:

$$737(X^{-1}) \times 456(X^1) = 336(X^1)+072(X^0)$$

Subtract:

$$\begin{array}{l} 336(X^1)+000(X^0)+000(X^{-1}) \\ -\,336(X^1)+072(X^0) \\ \hline 000(X^1)-072(X^0)+000(X^{-1}) \end{array}$$

Divide:

$$\frac{-072(X^0)+000(X^{-1})}{456(X^1)} = -158(X^{-2})$$

Continue in this way until you obtain the accuracy desired. The complete long-division array looks like this:

$$\begin{array}{r|l} & 270(X^0)+737(X^{-1})-158(X^{-2}) \\ \hline 456(X^1) & 123(X^2)+456(X^1)+000(X^0)+000(X^{-1})+000(X^{-2}) \\ & -\,123(X^2)+120(X^1) \\ & \quad 336(X^1)+000(X^0) \\ & \quad -\,336(X^1)+072(X^0) \\ & \qquad -072(X^0)+000(X^{-1}) \\ & \qquad -\,[-072(X^0)-048(X^{-1})] \\ & \qquad\qquad +048(X^{-1})+000(X^{-2}) \end{array}$$

The answer is found from the quotient:

$$(270 \times 1{,}000^0)+(737 \times 1{,}000^{-1}) - (158 \times 1{,}000^{-2})+(106 \times 1{,}000^{-3})$$

or, 270.736 842 106, with a small negative remainder. □

Engineer's Notebook is a regular feature in Electronics. We invite readers to submit original design shortcuts, calculation aids, measurement and test techniques, and other ideas for saving engineering time or cost. We'll pay $50 for each item published.